Solve the initial value problem $(\cos x)y' - (\sin x)y = xy^3\cos^4 x, \quad y(\pi) = -1$. **SCORE: _____ / 45 PTS**

$$y' - (\tan x)y = xy^3\cos^3 x \qquad \text{BERNOULLI}$$

$$v = y^{1-3} = y^{-2}$$

$$v' = -2y^{-3}y' \longrightarrow y' = -\tfrac{1}{2}y^3 v'$$

④ UNLESS OTHERWISE NOTED

$$-\tfrac{1}{2}y^3 v' - (\tan x)y = xy^3\cos^3 x$$

$$v' + (2\tan x)y^{-2} = -2x\cos^3 x$$

$$v' + (2\tan x)v = -2x\cos^3 x$$

$$\mu = e^{\int 2\tan x\,dx} = e^{-2\ln|\cos x|} = \sec^2 x$$

$$(\sec^2 x)v' + (2\sec^2 x\tan x)v = -2x\cos x$$

CHECK: $\frac{d}{dx}\sec^2 x = 2\sec x\sec x\tan x = 2\sec^2 x\tan x$ ✓

③ $$(\sec^2 x)v = \int -2x\cos x\,dx$$

$$= -2x\sin x - 2\cos x + C$$

u	dv
$-2x$	$+\cos x$
-2	$-\sin x$
0	$-\cos x$

$$y^{-2} = v = (-2x\sin x - 2\cos x + C)\cos^2 x$$

$$(-1)^{-2} = (0 + 2 + C)(1)$$

$$1 = 2 + C$$

③ $$C = -1$$

$$y^{-2} = (-2x\sin x - 2\cos x - 1)\cos^2 x$$

③ $$y = -\left[(-2x\sin x - 2\cos x - 1)\cos^2 x\right]^{-\frac{1}{2}}$$

SINCE $y(\pi) = -1$

An object moving at high speed may encounter air resistance proportial to the square of its velocity. Assume that $v > 0$ corresponds to downward motion.

SCORE: _____ / 15 PTS

[a] Write a differential equation for the velocity of an object rising at high speed.
Justify your answer briefly. All symbolic constants in your equation must represent positive numbers.

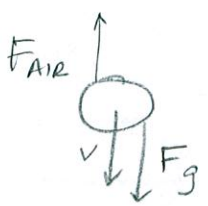
$m\frac{dv}{dt} = mg + kv^2$

1 2 $2\frac{1}{2}$

$m, g, k, v^2 > 0$

2 $F_{AIR}, F_g > 0$

[b] Write a differential equation for the velocity of an object falling at high speed.
Justify your answer briefly. All symbolic constants in your equation must represent positive numbers.



$m\frac{dv}{dt} = mg - kv^2$

1 2 $2\frac{1}{2}$

$m, g, k, v^2 > 0$

2 $F_{AIR} < 0, F_g > 0$

A population model which takes into account both the survival threshold and the environmental carrying capacity has a differential equation of the form $\frac{dP}{dt} = kP(P - P_S)(P_C - P)$,
where P_S is the survival threshold and P_C is the environmental carrying capacity, with $0 < P_S < P_C$.

SCORE: _____ / 20 PTS

[a] Determine the sign of k by considering the case in which the initial population is below the survival threshold.
Justify your answer briefly. (In fact, the sign of k is the same regardless of the initial population.)

$P_0 < P_s \rightarrow \frac{dP}{dt} < 0$ (GOING EXTINCT) AND $P - P_s < 0$

ALSO $P < P_s < P_c$, SO $P_c - P > 0$

AND $P > 0$ SO $k > 0$

[b] If the initial population is just slightly above the survival threshold, determine the population in the long run.
Justify your answer using techniques discussed in class.

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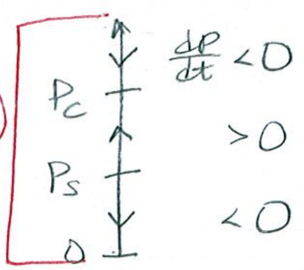

AS $t \rightarrow \infty$,
$P(t) \rightarrow P_c$

2 UNLESS OTHERWISE NOTED

[c] Determine if this model seems appropriate. **Justify your conclusion briefly.**

AS $t \rightarrow \infty$, $P \rightarrow 0$ OR P_c (IE. $P \not\rightarrow \infty$)

THE MODEL IS APPROPRIATE

The spread of a rumor is measured by how many people have heard of the rumor. The rate at which a rumor is spread is proportional to the number of encounters between people who have heard the rumor and people who have not heard the rumor. (This is similar to the predator-prey model, in which the rate at which prey are eaten is proportional to the number of encounters between predator and prey.) **SCORE: _____ / 35 PTS**

In a small town of 1000 people, the town gossip has learned of a new rumor (that no one else knows yet). After 5 days, half the town's population has heard of the rumor. Determine how many townspeople have heard of the rumor after t days. (Your final answer may use the symbol for the constant of proportionality, as long as you determine the value of the constant.)

P(t) = # PEOPLE WHO HAVE HEARD THE RUMOR AFTER t DAYS

$$\frac{dP}{dt} = kP(1000-P), \quad P(0)=1$$

$$\int \frac{1}{P(1000-P)}\,dP = \int k\,dt$$

3 UNLESS OTHERWISE NOTED

$$\int \left(\frac{\frac{1}{1000}}{P} + \frac{\frac{1}{1000}}{1000-P}\right) dP = kt + C$$

2

$$\frac{1}{1000}\left(\ln|P| - \ln|1000-P|\right) = kt + C$$

$$\ln\left|\frac{P}{1000-P}\right| = 1000kt + C$$

$$\frac{P}{1000-P} = Ce^{1000kt}$$

$$\frac{1000-P}{P} = Ce^{-1000kt}$$

$$\frac{1000}{P} - 1 = Ce^{-1000kt}$$

$$P = \frac{1000}{1+Ce^{-1000kt}}$$

$$P(0) = \frac{1000}{1+C} = 1$$

$$C = 999$$

2

$$P(5) = \frac{1000}{1+999e^{-5000k}} = 500$$

$$1+999e^{-5000k} = 2$$

2

$$k = -\frac{1}{5000}\ln\frac{1}{999}$$

OR

$$\frac{\ln 999}{5000}$$

$$P(t) = \frac{1000}{1+999e^{-\frac{\ln 999}{5}t}}$$

2

Solve the differential equation $y' = \dfrac{e^y - 2x}{2x^2 - 3xe^y}$.

SCORE: _____ / 35 PTS

$$\frac{dy}{dx} = \frac{e^y - 2x}{2x^2 - 3xe^y}$$

$$\underbrace{(e^y - 2x)}_{M}\,dx + \underbrace{(3xe^y - 2x^2)}_{N}\,dy = 0 \quad 4$$

$M_y = e^y \qquad N_x = 3e^y - 4x$

3 UNLESS OTHERWISE NOTED

$$\frac{N_x - M_y}{M} = \frac{2e^y - 4x}{e^y - 2x} = 2$$

$$\mu = e^{\int 2\,dy} = e^{2y} \quad 4$$

$$\underbrace{(e^{3y} - 2xe^{2y})}_{M}\,dx + \underbrace{(3xe^{3y} - 2x^2e^{2y})}_{N}\,dy = 0$$

$M_y = 3e^{3y} - 4xe^{2y} = N_x$ EXACT

$$f = \int (e^{3y} - 2xe^{2y})\,dx = xe^{3y} - x^2e^{2y} + C(y)$$

$$f_y = 3xe^{3y} - 2x^2e^{2y} + C'(y) = 3xe^{3y} - 2x^2e^{2y}$$

$$C'(y) = 0$$

$$C(y) = 0$$

$$xe^{3y} - x^2e^{2y} = C$$